Elimination Techniques In Modern Propositional Logic Reasoning

Norbert Manthey
nmanthey@conp-solutions.com

December 7, 2017

Outline

- Satisfiability Testing
- Elimination in SAT
  - Solving Algorithms
  - Constraint Types
  - Model Reconstruction
  - Variable Addition
- Conclusion

Satisfiability Testing

Propositional Logic

- Variables: $v_1, v_2, \dots \in \mathcal{V}$ of Boolean domain $\{\bot, \top\}$
  - often also seen as $\{0, 1\}$
- Connectives:
  - negation $\neg v_1$ (also written as $\overline{v_1}$)
  - disjunction $v_1 \vee v_2$
  - conjunction $v_1 \wedge v_2$
  - many more, can be defined over truth table
- Literals: $p, \neg q, x_1, \overline{x_2}, \dots$ are variables, or negated variables
  - double negation is eliminated
- Function $\text{vars}(F)$ returns set of variables of formula F
- Function $\text{lits}(F)$ returns set of literals of formula F

Propositional Logic - Semantics

- Interpretation: function that maps variables to truth values
  - total: map all variables of the input language
  - partial: map variables of the input language
  - complete (wrt. formula): map all variables of the formula
- An interpretation I **satisfies** a formula F, if the formula evaluates to $\top$ after mapping the variables to their truth values, i.e. $I \models F$.

Propositional Logic - Semantics

- Interpretation: function that maps variables to truth values
  - total: map all variables of the input language
  - partial: map variables of the input language
  - complete (wrt. formula): map all variables of the formula
- An interpretation I **satisfies** a formula F, if the formula evaluates to $\top$ after mapping the variables to their truth values, i.e. $I \models F$.
- A formula F is **satisfiable**, if such an interpretation I exists.
- **Satisfiability Testing**: Given a formula F, is it satisfiable?
  - Compute a model, an unsatisfiable subset or proof!

Propositional Logic - Conjunctive Normal Form (CNF)

- Proposition logic formulas can be complex
- Reasoners should be fast
- Pick reasonable subset

Propositional Logic - Conjunctive Normal Form (CNF)

- Proposition logic formulas can be complex
- Reasoners should be fast
- Pick reasonable subset

- Clause: disjunction of literals $(x_1 \vee \cdots \vee x_k)$
  - equal to a (multi)set of literals $\{x_1, \ldots, x_k\}$
- CNF Formula: conjunction of clauses $(C_1 \wedge \cdots \wedge C_n)$
  - equal to a (multi)set of clauses $\{C_1, \ldots, C_k\}$
- **Resolvent** of clauses C and D with $x \in C$ and $\overline{x} \in D$:
  - $C \otimes D = (C \setminus x) \cup (D \setminus \overline{x})$

Propositional Logic - Conjunctive Normal Form (CNF)

- Proposition logic formulas can be complex
- Reasoners should be fast
- Pick reasonable subset

- Clause: disjunction of literals $(x_1 \vee \cdots \vee x_k)$
  - equal to a (multi)set of literals $\{x_1, \ldots, x_k\}$
- CNF Formula: conjunction of clauses $(C_1 \wedge \cdots \wedge C_n)$
  - equal to a (multi)set of clauses $\{C_1, \ldots, C_k\}$
- **Resolvent** of clauses C and D with $x \in C$ and $\overline{x} \in D$:
  - $C \otimes D = (C \setminus x) \cup (D \setminus \overline{x})$
- **Reduct** F wrt set of literals x, $F|_x$: map x to $\top$, simplify
- **Subformula** F_x of F wrt literal x: clauses with x

Propositional Logic - Conjunctive Normal Form (CNF)

- Proposition logic formulas can be complex
- Reasoners should be fast
- Pick reasonable subset

- Clause: disjunction of literals $(x_1 \vee \cdots \vee x_k)$
  - equal to a (multi)set of literals $\{x_1, \ldots, x_k\}$
- CNF Formula: conjunction of clauses $(C_1 \wedge \cdots \wedge C_n)$
  - equal to a (multi)set of clauses $\{C_1, \ldots, C_k\}$
- **Resolvent** of clauses C and D with $x \in C$ and $\overline{x} \in D$:
  - $C \otimes D = (C \setminus x) \cup (D \setminus \overline{x})$
- **Reduct** F wrt set of literals x, $F|_x$: map x to $\top$, simplify
- **Subformula** F_x of F wrt literal x: clauses with x

$$F = \{\{x, y\}, \{\overline{x}, y\}\} \quad F|_x = \{\{y\}\} \quad F_x = \{\{x, y\}\}$$

Propositional Logic - Formula Relations

- Given, formulas F and G
- $F \models G$, if all (total) interpretations I with $I \models F$ also satisfy G, $I \models G$
- Equivalence $F \equiv G$: $F \models G$ and $G \models F$
- Equi-Satisfiability $F \equiv_{SAT} G$: F and G are both satisfiable, or F and G are both unsatisfiable
- **Unsatisfiability-Preserving** $F \models_{UNSAT} G$: if $F \models G$ and $F \equiv_{SAT} G$

Propositional Logic - Formula Relations

- Given, formulas F and G
- $F \models G$, if all (total) interpretations I with $I \models F$ also satisfy G, $I \models G$
- Equivalence $F \equiv G$: $F \models G$ and $G \models F$
- Equi-Satisfiability $F \equiv_{SAT} G$: F and G are both satisfiable, or F and G are both unsatisfiable
- **Unsatisfiability-Preserving** $F \models_{UNSAT} G$: if $F \models G$ and $F \equiv_{SAT} G$

$$x \models (x \vee y) \qquad x \equiv_{SAT} y$$
$$(x \wedge \overline{x}) \models y \qquad (x \wedge \overline{x}) \models_{UNSAT} (y \wedge \overline{y})$$

$(x \wedge \overline{x}) \models_{UNSAT} y$ does **not** hold!

Propositional Logic - Advanced Formula Relations

Definition (Model Constructibility)

A formula G is model constructible with respect to a formula F and to a set of variables S, in symbols $F \rightsquigarrow^{S}_{mc} G$, if for each total model I of F there exists a total model I' of G such that $I(x) = I'(x)$ for all $x \in (\mathcal{V} \setminus S)$.

Definition (Constructibility)

A formula G is constructible from a formula F, in symbols $F \rightsquigarrow_{\cap} G$, if for each model I of F there exists a model I' of G such that $I(x) = I'(x)$ for all $x \in \mathrm{vars}(F)$.

Definition (Mutual Constructibility)

Two formulas F and G are mutually constructible, in symbols $F \leftrightsquigarrow_{\cap} G$, if $F \rightsquigarrow_{\cap} G$ and $G \rightsquigarrow_{\cap} F$.

Mutual Constructibility

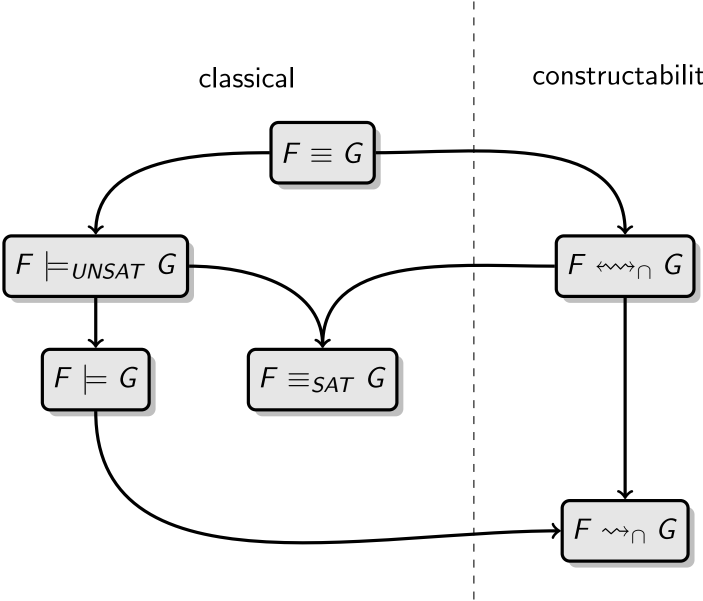
- Original formula
$$F = (x \vee d) \wedge (\overline{a} \vee \overline{b} \vee x) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x}) \wedge (\overline{x} \vee c)$$

- Formula without x, $\text{vars}(F) \cap \text{vars}(G) = \{a, b, c, d\}$
$$G = (d \vee a) \wedge (d \vee b) \wedge (\overline{a} \vee \overline{b} \vee c)$$

- Both satisfiable: $J_F = (abcdx)$ $\quad J_G = (abcd\overline{x})$

- By changing the mapping of x, J_F can be turned into J_G, and vice versa. In this example, $F \leftrightsquigarrow_{\cap} G$.

Formula Relations



More details in [Man14].

Elimination in SAT

Modern SAT Solving

- Successfully applied in different areas
  - hardware/software model checking, planning, optimization, verification, general purpose backend, ...
- Many different input pattern
  - AND-gates, XOR-gates, cardinality constraints, clauses
- Combine different solving strategies
- Special purpose techniques
  - Gaussian Elimination, Cardinality Extraction, Variable Elimination, Clause Eliminations, Variable Addition, Failed Literal Probing

Solving Algorithms

DavisPutnam (CNF formula F)

Input: A formula F in CNF

Output: The solution SAT or UNSAT of this formula

```
while true
   if F = ∅ then return SAT                          // satisfiability rule
   if ⊥ ∈ F then return UNSAT                        // unsatisfiability rule
   if (x) ∈ F then                                   // unit rule
      F := F|_x
      continue
   if x ∈ lits(F) and x̄ ∉ lits(F) then               // pure literal rule
      F := F|_x
      continue
   G := F \ {F_x ∪ F_x̄}                              // clauses without x
   F := G ∪ {F_x ⊗ F_x̄}                              // variable elimination
```

Using Elimination During Search

- **1960**: DP Algorithm [DP60]
- **1962**: search and backtracking instead of elimination (DLL) [DLL62]
- **1999**: backjumping and learning (CDCL) [MSS96]
- **200X**: improve heuristics, data structures [MMZ$^+$01, SE02]
- **2005**: (partial) variable elimination as preprocessing
  - MINISAT with SATELITE [EB05]
- **2009**: simplification during search [Bie09]
- **2009**: (partial) Gaussian elimination [SNC09]
- **2012**: automated variable addition [MHB13]
- **2013**: (partial) cardinality reasoning [BLBLM14]

- Systems like LINGELING, RISS or CRYPTOMINISAT implement most of the above and schedule heuristically.

(Bounded) Variable Elimination

- Formula F and variable v to be eliminated
- v might be **functionally dependent**, $v \leftrightarrow (a \wedge b)$
  - $G_v = \{(v \vee \overline{a} \vee \overline{b})\} \qquad G_{\overline{v}} = \{(\overline{v} \vee a), (\overline{v} \vee b)\}$
- before elimination, split:
  - $F_v = G_v \wedge R_v \qquad F_{\overline{v}} = G_{\overline{v}} \wedge R_{\overline{v}}$
- new clauses $S := F_v \otimes F_{\overline{v}}$
- if functional dependent $S := R_v \otimes G_{\overline{v}} \wedge G_v \otimes R_{\overline{v}}$

$$F' := (F \setminus (F_v \cup F_{\overline{v}})) \cup S$$

- Bounded (**number of clauses matters**):
  - $|S| \leq |F_v| + |F_{\overline{v}}|$, ignoring tautologies
  - $|F_v| \leq 5 \wedge |F_{\overline{v}}| \leq 15$, or symmetric

Variable Elimination Example

- Original formula

$$F = (x \vee d) \wedge (\overline{a} \vee \overline{b} \vee x) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x}) \wedge (\overline{x} \vee c)$$

Variable Elimination Example

- Original formula

$$F = (x \vee d) \wedge (\overline{a} \vee \overline{b} \vee x) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x}) \wedge (\overline{x} \vee c)$$

- Subformulas

Variable Elimination Example

- Original formula
$$F = (x \vee d) \wedge (\overline{a} \vee \overline{b} \vee x) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x}) \wedge (\overline{x} \vee c)$$

- Subformulas
$$G_x = (\overline{a} \vee \overline{b} \vee x) \qquad G_{\overline{x}} = (a \vee \overline{x}) \wedge (b \vee \overline{x})$$
$$R_x = (x \vee d) \qquad R_{\overline{x}} = (\overline{x} \vee c)$$

Variable Elimination Example

- Original formula

$$F = (x \vee d) \wedge (\overline{a} \vee \overline{b} \vee x) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x}) \wedge (\overline{x} \vee c)$$

- Subformulas

$$G_x = (\overline{a} \vee \overline{b} \vee x) \qquad G_{\overline{x}} = (a \vee \overline{x}) \wedge (b \vee \overline{x})$$
$$R_x = (x \vee d) \qquad R_{\overline{x}} = (\overline{x} \vee c)$$

- Formula without x

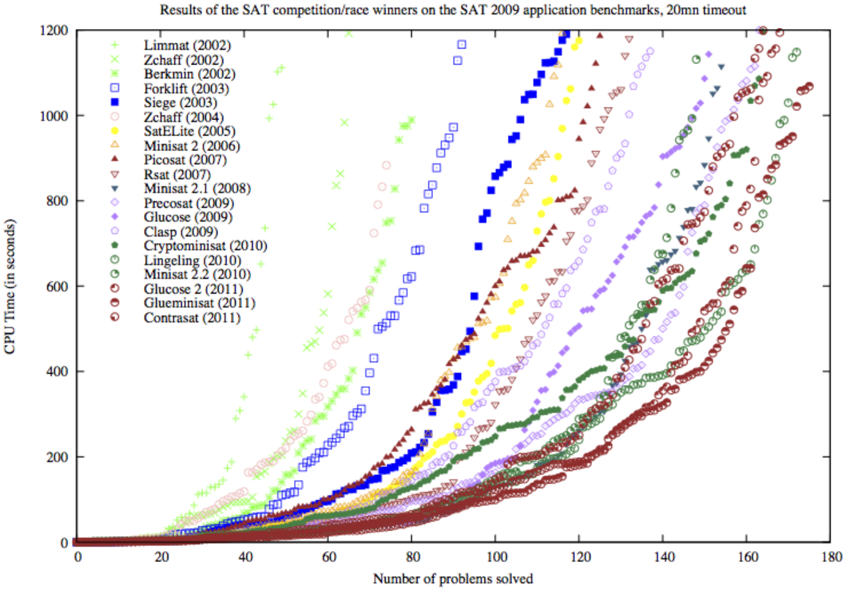
$$S := G_x \otimes R_{\overline{x}} \wedge R_x \otimes G_{\overline{x}}$$
$$S = (d \vee a) \wedge (d \vee b) \wedge (\overline{a} \vee \overline{b} \vee c)$$

- Redundant:

$$G_x \otimes G_{\overline{x}} = \top$$
$$R_x \otimes R_{\overline{x}} = (c \vee d)$$

BVE in 2005 won the competition significantly
(267 solved, 242 next)

Elimination using Constraints



(http://www.pragmaticsofssat.org/2012/application-cactus-pos12.png)

Elimination using Constraints

- Problems do not come in CNF
- F might contain cardinality constraints (CCs) or XORs
- Extract constraints, apply reasoning there
  - Boolean domain is $\{0, 1\}$ instead of $\{\bot, \top\}$
- Find new constraints to be encoded to CNF
  - or *efficiently* prove inconsistency

Elimination using Constraints

- Problems do not come in CNF
- F might contain cardinality constraints (CCs) or XORs
- Extract constraints, apply reasoning there
  - Boolean domain is $\{0, 1\}$ instead of $\{\perp, \top\}$
- Find new constraints to be encoded to CNF
  - or *efficiently* prove inconsistency
- Cardinality Constraints: $\sum_i w_i x_i \leq k$, with $w_i, k \in \mathbb{Z}$
  - Instead of resolution, use addition, and multiplication

Elimination using Constraints

- Problems do not come in CNF
- F might contain cardinality constraints (CCs) or XORs
- Extract constraints, apply reasoning there
  - Boolean domain is $\{0, 1\}$ instead of $\{\perp, \top\}$
- Find new constraints to be encoded to CNF
  - or *efficiently* prove inconsistency

- Cardinality Constraints: $\sum_i w_i x_i \leq k$, with $w_i, k \in \mathbb{Z}$
  - Instead of resolution, use addition, and multiplication

- XORs: $\sum_i x_i \bmod 2 = 1$, with $w_i, k \in \mathbb{Z}$
  - Instead of resolution, use addition with modulo
  - Find new XOR constraints to be encoded to CNF

Model Reconstruction

Model Reconstruction

- $J' \models F'$ does not imply $J' \models F$, v can be mapped arbitrarily
- solver only finds J'
- simplifier knows F

Model Reconstruction

- $J' \models F'$ does not imply $J' \models F$, v can be mapped arbitrarily
- solver only finds J'
- simplifier knows F

$$J = \begin{cases} (J' \setminus \{v\}) \cup \{\overline{v}\}, & \text{if } J' \not\models F_{\overline{v}} \\ (J' \setminus \{\overline{v}\}) \cup \{v\}, & \text{if } J' \not\models F_v \\ J', & \text{otherwise} \end{cases}$$

Model Reconstruction

- $J' \models F'$ does not imply $J' \models F$, v can be mapped arbitrarily
- solver only finds J'
- simplifier knows F

$$J = \begin{cases} (J' \setminus \{v\}) \cup \{\overline{v}\}, & \text{if } J' \not\models F_{\overline{v}} \\ (J' \setminus \{\overline{v}\}) \cup \{v\}, & \text{if } J' \not\models F_v \\ J', & \text{otherwise} \end{cases}$$

- Implementation
  - when eliminating v, store F_v and $F_{\overline{v}}$
  - or, store only F_v and set $J' := (J' \setminus \{v\}) \cup \{\overline{v}\}$

Variable Addition

Variable Addition

Definition (Extension)

A formula F with two literals l and l' that occur in F can be extended with a fresh variable x to $F' = F \land (x \lor l) \land (x \lor l') \land (\overline{x} \lor \overline{l} \lor \overline{l'})$.

- For any model J' with $J' \models F'$, also $J' \models F$
- What would happen when using variable elimination next?

- Used for short theoretical proofs (extended resolution)
  - There exists clause based short proofs for e.g. pigeon hole
- Cannot be automated efficiently (as far as we know)

Variable Addition

Definition (Extension)

A formula F with two literals l and l' that occur in F can be extended with a fresh variable x to $F' = F \wedge (x \vee l) \wedge (x \vee l') \wedge (\overline{x} \vee \overline{l} \vee \overline{l'})$.

- For any model J' with $J' \models F'$, also $J' \models F$
- What would happen when using variable elimination next?

- Used for short theoretical proofs (extended resolution)
  - There exists clause based short proofs for e.g. pigeon hole
- Cannot be automated efficiently (as far as we know)
- Exploit **number of clauses matters**?

(Bounded) Variable Addition BVA

- Can you reduce the number of clauses here?

  $F := (a \vee c) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee d) \wedge (b \vee e)$

(Bounded) Variable Addition BVA

- Can you reduce the number of clauses here?
  $F := (a \vee c) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee d) \wedge (b \vee e)$
- Simplified, with fresh variable x
  $F' := (x \vee c) \wedge (x \vee d) \wedge (x \vee e) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x})$

(Bounded) Variable Addition BVA

- Can you reduce the number of clauses here?
  $F := (a \vee c) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee d) \wedge (b \vee e)$
- Simplified, with fresh variable x
  $F' := (x \vee c) \wedge (x \vee d) \wedge (x \vee e) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x})$
- How about variable elimination on x?

(Bounded) Variable Addition BVA

- Can you reduce the number of clauses here?
  $F := (a \vee c) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee d) \wedge (b \vee e)$
- Simplified, with fresh variable x
  $F' := (x \vee c) \wedge (x \vee d) \wedge (x \vee e) \wedge (a \vee \overline{x}) \wedge (b \vee \overline{x})$
- How about variable elimination on x?
- BVA linearizes naive quadratic at-most-one encoding

Conclusion

Take Home Message

- Variable Elimination is an extremely powerful technique
- Produces mutual constructible formulas
- Similar techniques exist for higher level constraints
- The reverse – variable addition – is not that effective
- Elimination has to be applied limited

Elimination Techniques
In Modern Propositional Logic Reasoning

Norbert Manthey

nmanthey@conp-solutions.com

December 7, 2017

Thank you for your attention

Armin Biere.
PrecoSAT system description.
`http://fmv.jku.at/precosat/preicosat-sc09.pdf`, 2009.

Armin Biere, Daniel Le Berre, Emmanuel Lonca, and Norbert Manthey.
Detecting cardinality constraints in CNF.
In *SAT 2014*, volume 8561 of *LNCS*, pages 285–301, 2014.

Martin Davis, George Logemann, and Donald Loveland.
A machine program for theorem-proving.
Communications of the ACM, 5(7):394–397, 1962.

Martin Davis and Hilary Putnam.
A computing procedure for quantification theory.
Journal of the ACM, 7(3):201–215, 1960.

Niklas Eén and Armin Biere.
Effective preprocessing in SAT through variable and clause elimination.
In *SAT 2005*, volume 3569 of *LNCS*, pages 61–75, 2005.

Norbert Manthey.
Towards Next Generation Sequential and Parallel SAT Solvers.
PhD thesis, TU Dresden, 2014.

Norbert Manthey, Marijn J.H. Heule, and Armin Biere.
Automated reencoding of Boolean formulas.
In *Hardware and Software: Verification and Testing*, volume 7857 of *LNCS*, pages 102–117, 2013.

Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik.
Chaff: Engineering an efficient SAT solver.
In *DAC*, pages 530–535. ACM, 2001.

João P. Marques-Silva and Karem A. Sakallah.
GRASP – a new search algorithm for satisfiability.
ICCAD '96, pages 220–227. IEEE Computer Society, 1996.

Niklas Sörensson and Niklas Eén.
Minisat v1.13 - A SAT solver with conflict-clause minimization. 2005. SAT-2005 poster.
Technical report, Chalmers University of Technology, 2002.

Mate Soos, Karsten Nohl, and Claude Castelluccia.
Extending SAT solvers to cryptographic problems.
In *SAT 2009*, volume 5584 of *LNCS*, pages 244–257, 2009.