Cut Elimination and Second-Order Quantifier Elimination

Alessandra Palmigiano

7 December 2017
http://www.appliedlogictudelft.nl

Main questions

- Which logics have nice sequent calculi (are *properly displayable*)?
- If $\mathcal{L}$ is properly displayable, which of its axiomatic extensions are too?

Algorithmic correspondence theory can help in answering these questions.

Setting

Expansions of distributive lattice logic with normal modal operators of arbitrary arity and polarity type (**normal DLE-logics**).

Main results

- *Syntactic characterization* of properly displayable axiomatic extensions of basic normal DLE-logics.
- *Algorithmic computation* of structural rules corresponding to these additional axioms.

Analytic calculi and cut elimination

Main requirement for analyticity

Proofs must proceed by stepwise decomposition, without any insertion of information not contained in the conclusion:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, A \to B \vdash B}}{A \wedge (A \to B) \vdash B}$$

Core violating rule

$$\text{Cut}\ \dfrac{X \vdash {\color{red}A} \qquad {\color{red}A} \vdash Y}{X \vdash Y}$$

Cut elimination theorem (Gentzen, 1935)

If $X \vdash Y$ is derivable, then a derivation of $X \vdash Y$ exists in which *Cut* is never applied.

However, proofs of cut elimination theorem are:

- lengthy,
- error prone,
- non-modular.

Proper Display Calculi

Natural generalization of sequent calculi.
Sequents $X \vdash Y$, where X, Y are structures:
$A, \quad A ; B, \quad \ldots \quad X > Y, \quad \ldots$

structural symbols assemble and disassemble structures;
logical symbols assemble formulas.

Main feature: display property

$$\dfrac{\dfrac{\dfrac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

This machinery ensures the existence of a

Canonical proof of cut elimination

Canonical cut elimination

Theorem (Belnap 1982, Wansing 1997)

If a calculus satisfies the properties below, then it enjoys cut elimination and subformula property.

- C1: structures can disappear, formulas are **forever**;
- **tree-traceable** formula-occurrences, via suitably defined congruence:
  - C2: same shape, C3: non-proliferation, C4: same position;
- C5: **principal = displayed**;
- C6, C7: rules are closed under **uniform substitution** of congruent parameters;
- C8: **reduction strategy** exists when cut formulas are both principal.

Canonical cut elimination: proof strategy

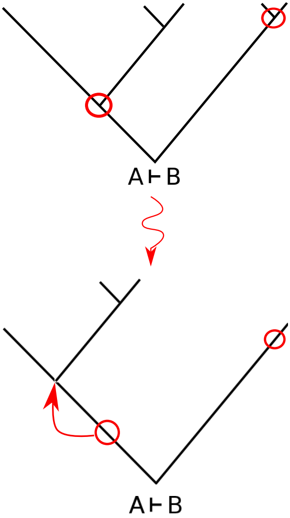
Principal stage

$$\dfrac{\dfrac{\vdots\,\pi_1 \atop Z \vdash \circ A}{Z \vdash \Box A} \qquad \dfrac{\vdots\,\pi_2 \atop A \vdash Y}{\Box A \vdash \circ Y}}{Z \vdash \circ Y}\ \mathit{Cut}$$

$$\Downarrow$$

$$\dfrac{\text{Display}\ \dfrac{\dfrac{\dfrac{\vdots\,\pi_1 \atop Z \vdash \circ A}{\bullet Z \vdash A} \qquad {\vdots\,\pi_2 \atop A \vdash Y}}{\bullet Z \vdash Y}\ \mathit{Cut}}{}}{Z \vdash \circ Y}\ \text{Display}$$

Parametric stage



Analytic structural rules

Those structural rules the shape of which supports the canonical cut elimination strategy:

- C1. structural variables in the premises appear also in the conclusion;
- C2. congruent parametric parts have the **same shape**;
- C3. structural variables in the premises are congruent to **at most one** variable in the conclusion (non-proliferation);
- C4. congruent parametric parts have the **same position** (either antecedent or consequent);
- C6,7. rules are closed under **uniform substitution** of congruent parameters;

Examples

$$\frac{X \vdash Y}{X \vdash Y \,;\, Z} \qquad \frac{X \,;\, X \vdash Y}{X \vdash Y} \qquad \frac{X \,;\, Z \vdash Y}{Z \,;\, X \vdash Y}$$

Non-Examples

$$\frac{X \vdash Y \,;\, Z}{X \vdash Y} \qquad \frac{X \vdash Y}{X \,;\, X \vdash Y} \qquad \frac{X \,;\, Z \vdash Y}{X < Z \vdash Y}$$

Normal DLE-logics

DLE: Distributive Lattice Expansions:
(distributive) lattice signature + operations of any finite arity.
Additional operations partitioned in families $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

Normality: In each coordinate,

- f-type operations *preserve* finite **joins** or *reverse* finite **meets**;
- g-type operations *preserve* finite **meets** or *reverse* finite **joins**.

Examples

- Distributive Modal Logic: $\mathcal{F} := \{\Diamond, \lhd\}$ and $\mathcal{G} := \{\Box, \rhd\}$
- Bi-intuitionistic modal logic: $\mathcal{F} := \{\Diamond, \succ\!\!-\}$ and $\mathcal{G} := \{\Box, \rightarrow\}$
- Full Lambek calculus: $\mathcal{F} := \{\circ\}$ and $\mathcal{G} := \{/, \backslash\}$
- Lambek-Grishin calculus: $\mathcal{F} := \{\circ, /_{\oplus}, \backslash_{\oplus}\}$ and $\mathcal{G} := \{\oplus, /_{\circ}, \backslash_{\circ}\}$
- ...

Relational/complex algebra semantics

- f-type operations have residuals $f_i^\sharp$ in each coordinate i;
- g-type operations have residuals $g_h^\flat$ in each coordinate h.

Proper display calculi for basic normal DLE-logics

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathsf{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Str.	I		;		>		H		K	
Log.	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\!\!-)$	$(\rightarrow)$	f			g

Str.	H_i		K_h	
Log.		$(f_i^\sharp)$	$(g_h^\flat)$	

  for $\varepsilon_f(i) = \varepsilon_g(h) = 1$

Str.	H_i		K_h	
log.	$(f_i^\sharp)$			$(g_h^\flat)$

  for $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

$$f_L \; \frac{H(A_1, \ldots, A_{n_f}) \vdash X}{f(A_1, \ldots, A_{n_f}) \vdash X} \qquad \frac{X \vdash K(A_1, \ldots, A_{n_g})}{X \vdash g(A_1, \ldots, A_{n_g})} \; g_R$$

$$f_R \; \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \quad \mid \quad \varepsilon_f(i) = 1 \quad \varepsilon_f(j) = \partial\right)}{H(X_1, \ldots, X_{n_f}) \vdash f(A_1, \ldots, A_{n_f})}$$

$$g_L \; \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \quad \mid \quad \varepsilon_g(i) = 1 \quad \varepsilon_g(j) = \partial\right)}{g(A_1, \ldots, A_{n_g}) \vdash K(X_1, \ldots, X_{n_g})}$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- If $\varepsilon_f(i) = \varepsilon_g(h) = 1$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{X_i \vdash H_i(X_1, \ldots, Y, \ldots, X_{n_f})} \qquad \frac{Y \vdash K(X_1 \ldots, X_h, \ldots X_{n_g})}{K_h(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h}$$

- If $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

$$\frac{H(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y}{H_i(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i} \qquad \frac{Y \vdash K(X_1, \ldots, X_h, \ldots, X_{n_g})}{X_h \vdash K_h(X_1, \ldots, Y, \ldots, X_{n_g})}$$

Primitive inequalities

Primitive formulas: [Kracht 1996]

$$\begin{array}{rl} \text{Left-primitive} & \varphi := p \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f(\vec{\varphi}/\vec{p}, \vec{\psi}/\vec{q}) \\ \text{Right-primitive} & \psi := p \mid \bot \mid \psi \wedge \psi \mid \psi \vee \psi \mid g(\vec{\psi}/\vec{p}, \vec{\varphi}/\vec{q}) \end{array}$$

Primitive inequalities:

Left-primitive: $\varphi_1 \leq \varphi_2$ with φ_1 scattered
(i.e. each p occurs at most once)

Right-primitive: $\psi_1 \leq \psi_2$ with ψ_2 scattered

Example: $\mathcal{F} := \{\Diamond\}, \mathcal{G} := \{\Box, \rightarrow\}$

Str.	$\circ$		$>$	
Log.	$\Diamond$	$\Box$		$\rightarrow$

$$\Diamond q \rightarrow \Box p \leq \Box(q \rightarrow p) \quad \rightsquigarrow \quad \frac{x \vdash \Diamond q \rightarrow \Box p}{x \vdash \Box(q \rightarrow p)} \quad \rightsquigarrow \quad \frac{X \vdash \circ Z > \circ Y}{X \vdash \circ(Z > Y)}.$$

Strategy

Crucial observation: **same** structural connectives for the **basic** and for the **expanded** DLE.

Main strategy: transform **non-primitive** DLE inequalities into (conjunctions of) **primitive** DLE inequalities in the **expanded** language:

$$s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q}) \qquad\qquad \& \left\{\varphi_i^*(\vec{p},\vec{q}) \leq {\varphi'}_i^*(\vec{p},\vec{q}) \mid i \in I\right\}$$

$$\Updownarrow \text{ ALBA} \qquad\qquad \Updownarrow \text{ ALBA on primitives}$$

$$\& \left\{\varphi_i^*(\vec{\boldsymbol{i}},\vec{\boldsymbol{m}}) \leq {\varphi'}_i^*(\vec{\boldsymbol{i}},\vec{\boldsymbol{m}}) \mid i \in I\right\} \qquad = \qquad \& \left\{\varphi_i^*(\vec{\boldsymbol{i}},\vec{\boldsymbol{m}}) \leq {\varphi'}_i^*(\vec{\boldsymbol{i}},\vec{\boldsymbol{m}}) \mid i \in I\right\}$$

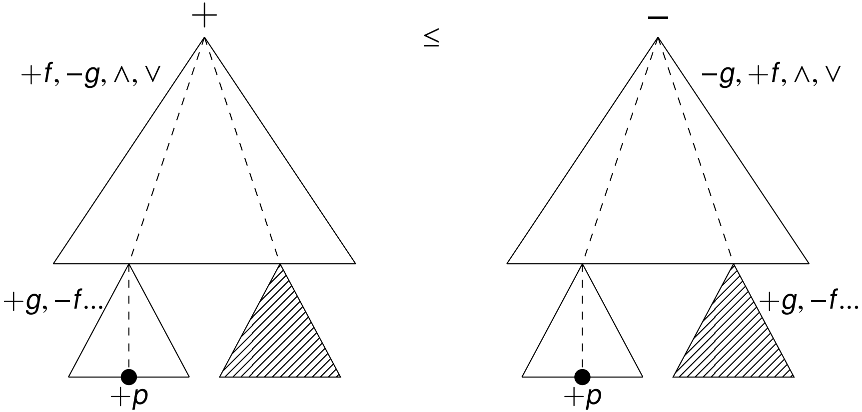
Inductive but not analytic

$$\forall[\Diamond p \leq \Diamond\Box p]$$

Inductive but not analytic

$$
\begin{array}{ll}
 & \forall[\Diamond p \leq \Diamond\Box p] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond p \ \& \ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq p \ \& \ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \Diamond\Box \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond\Box \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond\Box \mathbf{j}] \\
\text{iff} & \forall[\Diamond \mathbf{j} \leq \Diamond\Box \mathbf{j}]
\end{array}
$$

Analytic inductive inequalities



The Church-Rosser inequality

Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$.

$$\forall[\Diamond\Box p \leq \Box\Diamond p]$$

The Church-Rosser inequality

Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$.

$$
\begin{array}{ll}
 & \forall[\Diamond\Box p \leq \Box\Diamond p] \\
\text{iff} & \forall[\blacklozenge\Diamond\Box p \leq \Diamond p] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \mathbf{j} \leq \Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \blacklozenge\mathbf{j} \leq p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{i} \leq \blacklozenge\Diamond\mathbf{j} \ \&\ \Diamond\blacklozenge\mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall[\blacklozenge\Diamond\mathbf{j} \leq \Diamond\blacklozenge\mathbf{j}] \\
\hline
\text{iff} & \forall[\blacklozenge\Diamond p \leq \Diamond\blacklozenge p] \text{ (ALBA for primitive)}
\end{array}
$$

$$
\cdots \quad \rightsquigarrow \quad \frac{\Diamond\blacklozenge p \vdash z}{\blacklozenge\Diamond p \vdash z} \quad \rightsquigarrow \quad \frac{\circ \bullet X \vdash Z}{\bullet \circ X \vdash Z}
$$

ALBA-guided uniform strategy to derive the axiom from the rule

$$
\begin{array}{c}
p \vdash p \\
\hline
\Box p \vdash \circ p \\
\hline
\bullet \Box p \vdash p \\
\hline
\circ \bullet \Box p \vdash \Diamond p \\
\hline
\bullet \circ \Box p \vdash \Diamond p \\
\hline
\circ \Box p \vdash \circ \Diamond p \\
\hline
\circ \Box p \vdash \Box \Diamond p \\
\hline
\Diamond \Box p \vdash \Box \Diamond p
\end{array}
$$

The prelinearity axiom

Let $\mathcal{F} = \emptyset$, $\mathcal{G} = \{\rightharpoonup\}$ with $\rightharpoonup$ binary and of order-type $(\partial, 1)$.

$$\forall[\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)]$$

The prelinearity axiom

Let $\mathcal{F} = \varnothing$, $\mathcal{G} = \{\rightharpoonup\}$ with $\rightharpoonup$ binary and of order-type $(\partial, 1)$.

$$
\begin{array}{ll}
 & \forall[\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)] \\
\text{iff} & \forall[(r_1 \leq p \;\&\; q \leq r_2 \;\&\; r_3 \leq q \;\&\; p \leq r_4) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\
\text{iff} & \forall[(r_1 \leq r_4 \;\&\; q \leq r_2 \;\&\; r_3 \leq q) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\
\text{iff} & \forall[(r_1 \leq r_4 \;\&\; r_3 \leq r_2) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)]
\end{array}
$$

$$
\frac{X \vdash W \qquad Z \vdash Y}{I \vdash (X \succ Y) \,;\, (Z \succ W)}
$$

$$
\cfrac{\cfrac{p \vdash p \qquad q \vdash q}{\vdash (p \succ q) \,;\, (q \succ p)}}{\vdash (p \rightharpoonup q) \vee (q \rightharpoonup p)}
$$

Example

Let $\mathcal{G} = \varnothing$, $\mathcal{F} = \{\Diamond, \cdot\}$ where $\cdot$ binary and of order type $(1, 1)$

$$\forall[\Diamond\Diamond p \cdot \Diamond p \leq \Diamond p]$$

Example

Let $\mathcal{G} = \varnothing$, $\mathcal{F} = \{\Diamond, \cdot\}$ where $\cdot$ binary and of order type $(1, 1)$

$$
\begin{array}{ll}
 & \forall[\Diamond\Diamond p \cdot \Diamond p \leq \Diamond p] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond p \cdot \Diamond p \;\&\; \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond p \;\&\; \mathbf{i} \leq p \;\&\; \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \;\&\; \mathbf{i} \leq p \;\&\; \mathbf{h} \leq p \;\&\; \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \;\&\; \mathbf{i} \vee \mathbf{h} \leq p \;\&\; \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \;\&\; \Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \Rightarrow \forall \mathbf{m}[\Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{j} \leq \Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \Rightarrow \mathbf{j} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] \\
\text{iff} & \forall[\Diamond\Diamond \mathbf{i} \cdot \Diamond \mathbf{h} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] \\
\hline
\text{iff} & \forall[\Diamond\Diamond p_1 \cdot \Diamond p_2 \leq \Diamond p_1 \vee \Diamond p_2] \text{ (ALBA for primitive)}
\end{array}
$$

$$
\cdots \quad \rightsquigarrow \quad \frac{\Diamond p_1 \vdash q \quad \Diamond p_2 \vdash q}{\Diamond\Diamond p_1 \cdot \Diamond p_2 \vdash z} \quad \rightsquigarrow \quad \frac{\circ X \vdash Z \quad \circ Y \vdash Z}{\circ \circ X \odot \circ Y \vdash Z}
$$

Frege axiom: a first reduction

$$\forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)]$$

Frege axiom: a first reduction

$$
\begin{array}{ll}
 & \forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \,\&\, (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \,\&\, (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m} \,\&\, r \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \,\&\, (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \,\&\, (p \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \,\&\, \mathbf{i} \leq p) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \,\&\, (\mathbf{i} \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \,\&\, \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \,\&\, \mathbf{h} \leq \mathbf{i} \rightharpoonup q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \,\&\, \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \,\&\, \mathbf{i} \bullet \mathbf{h} \leq q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[(\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \,\&\, \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \forall \mathbf{m}[\mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] \\
\text{iff} & \forall[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})] \\
\text{iff} & \forall[\mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})] \\
\hline
\text{iff} & \forall[p \rightharpoonup ((p \bullet q) \rightharpoonup r) \leq q \rightharpoonup (p \rightharpoonup r)] \text{ (ALBA for primitive)}
\end{array}
$$

$$\vdots$$

$$\frac{\text{iff} \quad \forall[\mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})]}{\text{iff} \quad \forall[p \rightharpoonup ((p \bullet q) \rightharpoonup r) \leq q \rightharpoonup (p \rightharpoonup r)] \text{ (ALBA for primitive)}}$$

by applying the usual procedure, we obtain the following rule:

$$\cdots \quad \rightsquigarrow \quad \frac{s \vdash p \rightharpoonup ((p \bullet q) \rightharpoonup r)}{s \vdash q \rightharpoonup (p \rightharpoonup r)} \quad \rightsquigarrow \quad \frac{W \vdash X \succ ((X \odot Y) \succ Z)}{W \vdash Y \succ (X \succ Z)}$$

Frege axiom: a second reduction

$$\forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)]$$

Frege axiom: a second reduction

$$
\begin{array}{ll}
 & \forall[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] \\
\text{iff} & \forall[(p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q) \leq p \rightharpoonup r] \\
\text{iff} & \forall[((p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q)) \bullet p \leq r] \\
\text{iff} & \forall[\boldsymbol{i} \leq ((p \rightharpoonup (q \rightharpoonup r)) \bullet (p \rightharpoonup q) \bullet p \;\&\; r \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[\boldsymbol{i} \leq (\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \;\&\; \boldsymbol{h} \leq p \rightharpoonup (q \rightharpoonup r) \;\&\; \\
 & \qquad \boldsymbol{k} \leq p \rightharpoonup q \;\&\; \boldsymbol{j} \leq p \;\&\; r \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[\boldsymbol{i} \leq (\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \;\&\; (\boldsymbol{h} \bullet p) \bullet q \leq r \;\&\; \\
 & \qquad \boldsymbol{k} \bullet p \leq q \;\&\; \boldsymbol{j} \leq p \;\&\; r \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[\boldsymbol{i} \leq (\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \;\&\; (\boldsymbol{h} \bullet \boldsymbol{j}) \bullet q \leq r \;\&\; \boldsymbol{k} \bullet \boldsymbol{j} \leq q \;\&\; r \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[\boldsymbol{i} \leq (\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \;\&\; (\boldsymbol{h} \bullet \boldsymbol{j}) \bullet (\boldsymbol{k} \bullet \boldsymbol{j}) \leq r \;\&\; r \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[\boldsymbol{i} \leq (\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \;\&\; (\boldsymbol{h} \bullet \boldsymbol{j}) \bullet (\boldsymbol{k} \bullet \boldsymbol{j}) \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}] \\
\text{iff} & \forall[(\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \leq (\boldsymbol{h} \bullet \boldsymbol{j}) \bullet (\boldsymbol{k} \bullet \boldsymbol{j})] \\
\hline
\text{iff} & \forall[(r \bullet q) \bullet p \leq (r \bullet p) \bullet (q \bullet p)] \text{ (ALBA for primitive)}
\end{array}
$$

$$\vdots$$

$$\text{iff} \quad \forall[(\boldsymbol{h} \bullet \boldsymbol{k}) \bullet \boldsymbol{j} \leq (\boldsymbol{h} \circ \boldsymbol{j}) \bullet (\boldsymbol{k} \bullet \boldsymbol{j})]$$

$$\text{iff} \quad \forall[(r \bullet q) \bullet p \leq (r \bullet p) \bullet (q \bullet p)] \text{ (ALBA for primitive)}$$

by applying the usual procedure, we obtain the following rule:

$$\cdots \quad \rightsquigarrow \quad \frac{(r \bullet p) \bullet (q \bullet p) \vdash s}{(r \bullet q) \bullet p \vdash s} \quad \rightsquigarrow \quad \frac{(Z \odot X) \odot (Y \odot X) \vdash W}{(Z \odot Y) \odot X \vdash W}$$

Properties of rules and calculi guaranteed by ALBA

- The analytic structural rule ALBA-corresponding to a given inequality is **sound** on the class of algebras/frames defined by that inequality.
- ALBA runs on analytic inductive inequalities encode instructions for the cut-free derivations of the same inequality using the analytic structural rule(s) corresponding to it. Hence the resulting calculus is **syntactically complete** w.r.t. the corresponding Hilbert-style logic.
- Analytic inductive inequalities are canonical. Hence, the resulting calculus is a **conservative extension** of the corresponding Hilbert-style logic.
- **Cut elimination** and **subformula property** are guarantee by the general theory of proper display calculi.

Conclusions

- There are surprising connections between algorithmic correspondence theory and structural proof theory, seminally observed by Kracht.
- The same algorithm ALBA originally introduced to compute the first order correspondent of DLE-formulas and inequalities can be used to compute the analytic structural rule(s) corresponding to analytic inductive inequalities.
- Analytic structural rules have been identified as exactly those supporting the canonical strategy for cut elimination for proper display calculi.
- Analytic inductive inequalities exactly correspond to analytic structural rules.
- ALBA guarantees that the resulting analytic calculus is sound, complete and conservative.